Graph Coloring

Discrete Mathematics (MA 2333)
Faculty of Science
Telkom Institute of Technology
Bandung - Indonesia
Graph Coloring

- Graph coloring has variety of applications to Optimization problems, such as scheduling, frequency assignments problems, index register in a compiler, and so on
- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color (In addition to vertex coloring, there are edge coloring and region coloring)
- A graph can be colored by assigning a different color to each of its vertices. However, for most graphs a coloring can be found that uses fewer colors than number of vertices.
- What is the least number of colors necessary?
The chromatic number of a graph $G$ is the smallest number of colors with which it can be colored.

Notation: $\chi(G)$

In the example above, the chromatic number is 4.
Planar Graph Coloring

- The four color theorem: For every planar graph, the chromatic number is ≤ 4
Welch-Powell Algorithm

- Welch-Powell algorithm is an efficient algorithm for coloring graph $G$. This algorithm only gives us an upper bound of the chromatic number of $G$.
- Thus, this algorithm does not always give the number of minimum color needed in graph coloring.
Welch-Powell Algorithm

- Sort all vertices on graph $G$ based on their degree starting from the biggest to the smallest. This order is not unique because several vertices perhaps have similar degree.
- Use first color to color first vertex and other vertex which is on the order as long as the vertex is not adjacent to the previous step.
- Give second color to color vertex on the highest order (which has not been colored). Do it just like the previous step.
- It is like the third step. Do it continuously until every vertex on the graph becomes colorful.
Welch-Powell Algorithm

Use Welch Powell algorithm for graph coloring

Thus $\chi(H) = 4$
Welch-Powell Algorithm

Use Welch Powell algorithm for graph coloring

<table>
<thead>
<tr>
<th>Vertices</th>
<th>V1</th>
<th>V6</th>
<th>V2</th>
<th>V3</th>
<th>V4</th>
<th>V5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Color</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>c</td>
<td>c</td>
</tr>
</tbody>
</table>

Thus $\chi(G) = 3$
Bipartite Graph Coloring

- The chromatic number of bipartite is 2
- A simple graph G is called bipartite graph if its vertex set V can be partitioned into two disjoint subsets \( V_1 \) and \( V_2 \) (see previous section)
- Vertices with same color are in the same subset
Bipartite Graph Coloring

Vertices: v1, v2, v3, v4, v5, v6
Degree: 4, 4, 2, 2, 2, 2
Color: a, a, b, b, b, b

Subset 1: {v1, v2}
Subset 2: {v3, v4, v5, v6}
Bipartite Graph Coloring

Graph H

<table>
<thead>
<tr>
<th>Vertices</th>
<th>V1</th>
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<th>V3</th>
<th>V4</th>
<th>V5</th>
<th>V6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Color</td>
<td>a</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

Jadi $\chi(H) = 2$

subset 1 : \{v1, v4, v5\}
subset 2 : \{v2, v3, v6\}
Map Coloring (Region Coloring)

- Region Coloring can be done only on planar graph/plane graph
- Region coloring/map coloring is the assignment of a color to each region of the planar graph so that no two adjacent region are assigned the same color
- Two regions on a planar graph are said to be adjacent if they have least one common border
Map Coloring (Region Coloring)

- r1 adjacent with r2, r4, r6
- r2 adjacent with r1, r3
- r5 adjacent with r3
- r6 adjacent with r1, r3, r4
Map Coloring (Region Coloring)

r1 : green
r2 : red
r3 : blue
r4 : red
r5 : green
r6 : blue

Chromatic Number = 3
Map Coloring (Region Coloring)

Map coloring on a real map
Dual Graph

- Relationship between map coloring – vertex coloring
  - Each region of the map (plane graph) is represented by a vertex
  - Edges connect two vertices if the regions represented by these vertices have a common border
- The resulting graph is called the dual graph of the map (plane graph)
Dual Graph

Given graph $G$

Dual graph of $G$
Dual Graph

- vertex coloring by Welch Powell algorithm on this dual graph

<table>
<thead>
<tr>
<th>Vertices</th>
<th>v1</th>
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<th>v4</th>
<th>v5</th>
<th>v6</th>
</tr>
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<td>4</td>
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<tr>
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<td>a</td>
<td>c</td>
</tr>
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Chromatic Number = 3
Dual Graph

Given graph H

Dual graph of H
**Dual Graph**

- vertex coloring by Welch Powell algorithm on this dual graph

<table>
<thead>
<tr>
<th>vertices</th>
<th>v4</th>
<th>v8</th>
<th>v1</th>
<th>v2</th>
<th>v5</th>
<th>v7</th>
<th>v3</th>
<th>v6</th>
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</thead>
<tbody>
<tr>
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<td>2</td>
<td>1</td>
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<tr>
<td>color</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>c</td>
<td>b</td>
</tr>
</tbody>
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Chromatic Number = 4
Dual Graph

On the other hand, we can assign the colors to graph \( H \) like this:

We can see that the chromatic number is 3.

By Welch Powell algorithm, we have got the chromatic number is 4.

WHY?
Applications of Graph Coloring

- Scheduling Final Exams

Suppose there are 7 finals to be scheduled. Suppose the courses are numbered 1 through 7. Suppose that the following pairs of courses have common students:

1 and 2, 1 and 3, 1 and 4, 1 and 7, 2 and 3, 2 and 4, 2 and 5, 2 and 7, 3 and 4, 3 and 6, 3 and 7, 4 and 5, 4 and 6, 5 and 6, 5 and 7, 6 and 7

How can the final exams at a university be scheduled so that no student has two exams at the same time?
Applications of Graph Coloring

Chromatic number = 4
Thus, four time slots needed:
Time slot I : 2, 6
Time slot II : 3, 5
Time slot III : 4, 7
Time slot IV : 1

<table>
<thead>
<tr>
<th>vertices</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>7</th>
<th>1</th>
<th>5</th>
<th>6</th>
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Exercises

- Determine the chromatic number of graphs below
Exercises

Graphs below are bipartite graphs. Determine two subsets of vertices for each graph!

- For the graph on the left:
  - Subset 1: \{v1, v3, v5, v7\}
  - Subset 2: \{v2, v4, v6, v8\}

- For the graph on the right:
  - Subset 1: \{A, C, E\}
  - Subset 2: \{B, D, F, G\}
Exercises

- Twelve faculty members in an informatics department serve on the following committees:
  - *Undergraduate education*: Sineman, Limitson, Axiomus, Functionini
  - *Graduate Education*: Graphian, Vectorades, Functionini, Infinitescu
  - *Colloquium*: Lemmeau, Randomov, Proofizaki
  - *Library*: Van Sum, Sineman, Lemmeau
  - *Staffing*: Graphian, Randomov, Vectorades, Limitson
  - *Promotion*: Vectorades, Van Sum, Parabolton

- The committees must all meet during the first week of classes, but there are *only three time slots* available. Find a schedule that will allow all faculty members to attend the meetings of all committees on which they serve.
References
